Three-Dimensional Fourier Fringe Analysis and Phase Unwrapping

Hussein Abdul-Rahman

A thesis submitted in partial fulfilment of the requirements of Liverpool John Moores University for the degree of Doctor of Philosophy

General Engineering Research Institute (GERI), Liverpool John Moores University.

June 2007
“Allah will exalt in degree those of you who believe and those who have been granted knowledge. And Allah is well-acquainted with what you do”

Holy Quran

Chapter 58, Verse 11

The Prophet Mohammad, peace be upon him, Said “Whoever takes a path in search for knowledge, Allah will facilitate for him a path to Paradise”

Prophet Teachings
Dedicated to:
My beloved parents
Three-Dimensional Fourier Fringe Analysis and Phase Unwrapping

Abstract

Three-Dimensional Fourier Fringe Analysis and Phase Unwrapping

Hussein Sudqi Abdul-Rahman
Ph.D. Thesis

For many years the two-dimensional Fourier Fringe Analysis (FFA) technique has been regarded as a fast and reliable technique for the analysis of fringe patterns projected onto static objects. Today, two-dimensional FFA is seen as a fast and flexible method for processing fringe patterns of a dynamic object. But it is still inherently a two-dimensional approach, i.e. it deals with three-dimensional data (a video sequence) on the basis of regarding the data-set as a series of individual 2D images. The analysis of each 2D image is performed completely independently, no information from the previous images, or following images, is available at the time of processing the current image.

In the case of dynamic objects, we need new powerful techniques capable of processing the whole video sequence at once, instead of seeing it as a series of disconnected two-dimensional images. Regarding the data as a single unit means involving all variations of the fringe pattern, in space and time, in the processing procedure. In other words, three-dimensional processing may give us great potential benefits by taking into account the time variation of the fringe pattern, which was previously ignored.

The extension of the two-dimensional FFA technique into three-dimensions requires the extension of the current two-dimensional phase unwrapping algorithms into three-dimensional form. Phase unwrapping can be simply defined to be the process of solving the ambiguity problem caused by the fact that the absolute phase is typically wrapped into the interval $(-\pi, \pi)$. Phase unwrapping is considered to be a real challenge in fringe pattern analysis and many other applications, even in its two-dimensional form.

In this thesis, a novel three-dimensional FFA system has been implemented and used for demodulating fringe pattern sequences of dynamic objects. In addition, this thesis illustrates two novel three-dimensional phase unwrapping algorithms. The first algorithm attempts to find the best unwrapping path in the three-dimensional wrapped-phase volume. The second algorithm follows a best path approach to unwrap the phase volume, but takes into account the effect of singularity loops. Singularity loops are defined here as the source of noise that must be avoided during unwrapping. The two different algorithms have been tested on both simulated and real objects. The results show outstanding performance for these algorithms when unwrapping fringe volumes with very high levels of noise. This thesis also compares the performance of the proposed algorithms with other existing two-dimensional and three-dimensional phase unwrapping algorithms.
Three-Dimensional Fourier Fringe Analysis and Phase Unwrapping

Acknowledgement

I would like to thank Doctor Munther Gdeisat for the intellectual support, encouragement, continuing guidance and motivation, modesty, friendship and good humour throughout, which made this thesis possible, and for his patience in correcting both my stylistic and scientific errors. Thanks Munther.

I would like to express my deep gratitude to Professor Michael Lalor and Professor David Burton for their professional guidance, inspiration, invaluable advice, patience, friendly supervision, constant support and encouragement throughout the different stages of the project. I also wish to thank Doctor Francis Lilley for his support and help in editing and correcting this thesis. Thanks for everything.

I am also grateful to Bashar Rajoub, Abdulbasit Abid, Salah Karout, Mahmoud Al-Ghreify, Mohammad AIS’ad and all other colleagues for their advice, interesting discussions and helpful suggestions and for the good time I spent with them.

A special thank to my parents for their endless support throughout my studies. Without their constant assurance and assistance, completion of this project would have not been possible. Their enthusiasm, wisdom, drive, strength of character, tenacity and determination have inspired me to carry out this research work. And as a sign of my love, gratitude and affection I dedicate this work to them. I want to thank my lovely wife for her support and motivation in the completion of my PhD degree. Also, I would like to thank all of my brothers and sisters for their help and encouragement.

Finally, I acknowledge Doctor Mohammad Bataineh for his support and help during the early stages of this research.

Thank you all.

Hussein Abdul-Rahman
Table of Contents

Abstract ................................................................................. i
Acknowledgement ............................................................... ii
Table of Contents ............................................................... iii

1. Introduction ................................................................. 1
   1.1. Introduction ............................................................. 2
   1.2. Fringe Projection and Analysis ...................................... 4
   1.3. Phase Unwrapping and its Applications ......................... 8
   1.4. Research Objective ................................................... 10
   1.5. Contributions .......................................................... 10
   1.6. Thesis Structure ....................................................... 11
References ........................................................................... 13

2. Fringe Pattern Analysis Techniques ................................. 16
   2.1. Introduction ............................................................. 17
   2.2. Direct Phase Demodulation ......................................... 17
   2.3. Phase Stepping ........................................................ 23
   2.4. Fourier Transform Fringe Analysis ............................... 28
   2.5 Wavelet Transform Fringe Analysis ............................... 35
   2.6. Summary ................................................................. 38
References ........................................................................... 39

3. A Review of Two-Dimensional Phase Unwrapping Algorithms 42
   3.1. Introduction ............................................................. 43
   3.2. Local Phase Unwrapping Algorithms ......................... 44
      3.2.1. Quality-Guided Phase Unwrapping Algorithms ....... 48
Chapter One

Introduction
Chapter One

Introduction

1.1 Introduction

Optical non-contact measurement methods are very effective techniques for measuring the height of an object without affecting its surface. Indeed, optical measurements play a much more important role today, in the field of metrology, than they ever did in the past. More precise optical equipment, very fast computers and reliable image processing software increase the total reliability and accuracy of the measurement system, making optical metrology the first choice across many engineering and scientific disciplines, such as, medical applications (Woisetschlager et al., 1994; Lilley, 1999; Lilley et al., 2000; Engelsman et al., 2003; Moore et al., 2003), engineering and industrial applications (Kujawinska and Sitnik, 2000; Haist and Tiziani, 2002; Leopold et al., 2003; Quan et al., 2003), robot and machine vision (Tao-Xian and Xianyu-Su, 2001; Chen and Li, 2003; Smith and Smith, 2003) and in many other applications.

One of the most effective non-contact measurement techniques involves the use of structured lighting patterns, which are projected onto the object’s surface to obtain its 3D height. Perhaps the commonest form of structured-light is that of projected fringe patterns. The different methods used to demodulate fringe patterns in order to calculate 3D object height are referred as fringe pattern analysis techniques. Several algorithms have been proposed to analyse these fringe patterns, including phase stepping techniques (Chan et al., 1995), Fourier fringe analysis (FFA) (Takeda et al., 1982; Bone et al., 1986), direct phase detection (DPD) (Ichioka and Inuiya, 1972), wavelet transform fringe analysis (Dursun et al., 2004; Abid et al., 2006; Gdeisat et al., 2006) and many other algorithms.
Takeda’s original implementation of Fourier fringe analysis was one-dimensional, in that it only analysed a single line of the image at a time. However, it was not very long before the technique was extended into two-dimensions (Bone et al., 1986; Gorecki, 1992) and, it has undergone considerable development over time (Burton and Lalor, 1989a; Burton and Lalor, 1989b; Burton and Lalor, 1994; Burton et al., 1995; Su and Chen, 2001; Su et al., 2001).

For many years the two-dimensional Fourier transform fringe analysis technique has been regarded as being a fast and reliable technique for the analysis of fringe patterns projected onto static objects. Today, two-dimensional Fourier fringe analysis is seen as a fast and flexible method for processing fringe patterns of a dynamic object. But it is still inherently a two-dimensional approach, i.e. it deals with three-dimensional data (a video sequence) on the basis of an unconnected series of individual two-dimensional images. The analysis of each image is completely independent, no information from the previous images, or following images, is available at the time of processing the current image. Consequently, the use of this technique to analyse a sequence of fringe patterns for a dynamic object does not utilise any relationship between consecutive fringe patterns.

In this thesis, the author proposes a novel approach to analyse dynamic fringe pattern video sequences. The proposed technique extends 2D-FFA into three dimensions. This new proposed algorithm analyses the sequence of fringe patterns together as an entire volume, not as a series of individual images. This has the advantage of making the algorithm more robust, accurate and suitable for the measurement of dynamic objects when compared to the 2D-FFA algorithm.

Any of the Fourier fringe analysis algorithms can be divided into two main processing stages: phase extraction and phase unwrapping. The first stage extracts the phase of a fringe pattern by using a Fourier transform and carrying out filtering in the frequency domain. The phase produced by the first stage is wrapped and is typically limited to the $-\pi$ and $\pi$ range. The wrapped phase contains $2\pi$ jumps, which should be removed by using a phase unwrapping algorithm. This constitutes the second main stage in any FFA
algorithm. On that basis, the extension of the 2D-FFA into three dimensions requires the extension of both stages into 3D, as will be shown in this thesis.

The suitability of the algorithm for measuring dynamic objects could benefit many applications. For example, in medicine the 3D-FFA algorithm could be used for the dynamic monitoring of patients during radiotherapy treatment (Lilley et al., 2000).

1.2 Fringe Production and Analysis

Fringe Analysis techniques (Rastogi, 1997) are considered to be an effective, reliable and robust optical non-contact method for measuring 3D surface height. In these methods; a structured lighting pattern is projected onto the object’s surface. Due to the surface shape of the object, the projected pattern will be modified according to the object’s 3D height. This pattern is typically captured by a CCD camera and then stored in computer memory by the use of a frame-grabber. The image is then analysed by one of a range of fringe analysis algorithms. Finally, the phase is extracted and related to actual object height by use of some sort of phase-height relationship that has been previously determined during system calibration. This type of typical fringe projection system is shown in Figure 1.1.

In the case of dynamic objects, the CCD camera captures a sequence of fringe patterns. This sequence of fringe patterns represents the 3D movement of the object during the measurement period.

Fringes can be produced by using laser interferometers or by using digital projectors. In this project, laser interferometers were used to generate fringe patterns, as shown in Figure 1.1. As shown in the figure, two coherent laser beams projected from a twin fibre interferometer are used to illuminate an object. These beams will interfere with each other to produce a fringe pattern which is modulated by the object’s surface. This pattern encodes the height profile of the object’s surface. This pattern is captured by a camera to be analysed using a computer by means of fringe analysis algorithms.
The twin-fibre optical system is shown in Figure 1.2. As shown in the figure, the generated laser beam will propagate into an optic fibre cable until it reaches an optical coupler. This optical coupler will divide the laser beam into two beams. These two beams will propagate into two different fibres and then illuminate the object as shown in the figure. The twin-fibre interferometer is mainly used throughout this research to project fringes on objects.
One of the twin optical fibres can be moved in slight steps by a computer-controlled mechanism. A translation of the optical fibre along a line connecting the two fibre ends varies the spacing between fringes ($P_o$).

In the setup shown in Figure 1.1, a fringe pattern is produced by interfering two coherent laser beams with each other. When the two laser waves are superposed, the resultant intensity at any point depends on whether these waves reinforce or cancel each other. If the two waves have the same phase, i.e., the phase difference is zero, the intensity will be maximised, and if the two waves are $180^\circ$ out of phase, then the intensity will be minimised, as shown in Figure 1.3. Repeating occurrences of the maximum and the minimum phase differences will produce a repeating pattern of maximum and minimum intensity forming what is called a fringe pattern as shown in Figure 1.3.

![Figure 1.3: Fringe pattern for a reference plane.](image)

However, the previous fringe pattern shown in Figure 1.3 will be modified in the presence of an object, when viewed obliquely. The Fringe pattern will be modulated according to the object’s 3D height and the angle $\theta$ between the illumination and viewing axes as shown in Figure 1.4. Accordingly, the height profile of the object under examination is encoded as a function of the spatial phase of the fringe pattern ($\varphi$) that has been projected onto the object’s surface. In other words, the object’s height will
modulate the intensity distribution of the fringe pattern as expressed in the following equation (Robinson and Reid, 1993):

\[
g(x, y) = a(x, y) + b(x, y) \cos(2\pi f_o x + \varphi(x, y))
\] (1.1)

Where \( a(x, y) \) represents the background illumination, \( b(x, y) \) is the amplitude modulation of fringes, \( f_o \) is the spatial carrier frequency, \( \varphi(x, y) \) is the phase modulation of fringes (the required phase distribution) and \( x \) & \( y \) the sample indices for the \( x \) and \( y \) axes respectively.

Figure 1.4: Fringe pattern for an object.

The task now is to extract the phase \( \varphi \) from the intensity distribution of the fringe pattern, given in Equation (1.1), using a fringe analysis technique. Many fringe analysis techniques have been proposed to demodulate fringe patterns. These techniques are reviewed in chapter two of this thesis. Unfortunately, these techniques produce what is called wrapped-phase \( \psi \) instead of the required phase \( \varphi \). Consequently, phase unwrapping algorithms are required to recover the true phase \( \varphi \) from the wrapped-phase \( \psi \). Phase unwrapping algorithms are discussed in chapter three of this thesis. Finally, the calculated phase difference \( \varphi \) has to be converted into the appropriate
height using a reliable phase to height model (Rajoub et al., 2005). The steps of fringe patterns analysis can be summarized as in Figure 1.5.

![Fringe pattern analysis for an object](image)

**Figure 1.5: Fringe pattern analysis for an object.**

### 1.3 Phase Unwrapping and its Application

Phase unwrapping is a technique used on wrapped phase images to remove the $2\pi$ discontinuities embedded within the phase map. It detects a $2\pi$ phase jump and adds or subtracts an integer offset of $2\pi$ to successive pixels following that phase jump based on a threshold mechanism, thus, retrieving the continuous form of the phase map.

Phase unwrapping has many applications in many imaging technologies such as: optical and interferometric imaging, magnetic resonance imaging (MRI), synthetic aperture radar (SAR), synthetic aperture sonar, adaptive optics, seismic processing and aperture synthesis radio astronomy (Ghiglia and Pritt, 1998). In many of the applications mentioned the extracted phase relates to physical quantities, for example surface height in interferometry, wave-front distortion in adaptive optics, the degree of magnetic field inhomogeneity in the water/fat separation problem of magnetic resonance imaging or in the relationship between object phase and its associated bi-spectrum phase in astronomical imaging.

Techniques such as these, where the required information is encoded in phase form all require the use of a phase unwrapping algorithm. Figure 1.6 shows some wrapped-phase
and unwrapped-phase images that have been produced from a range of different applications.

Figure 1.6: Importance of phase unwrapping in many applications. The wrapped and unwrapped phase maps for: (Row 1) a human head from MRI image (Ghiglia and Pritt, 1998), (Row 2) part of the earth obtained from SAR image (Ghiglia and Pritt, 1998), (Row 3) a human breast undergoes radiotherapy treatment and (Row 4) artistic statue of a fairy. (Column a) wrapped phase maps and (Column b) corresponding unwrapped phase maps for those wrapped maps shown in column a.
1.4 Research Objectives

The aim of this project is to investigate the technical issues and potential advantages in implementing three-dimensional Fourier fringe analysis and three-dimensional phase unwrapping, with particular respect to the analysis of dynamic objects.

The objectives of this research can be summarised as follows.

- Developing and implementing a three-dimensional Fourier fringe analysis technique. This step includes the development of three-dimensional windows for the sequence of fringe patterns, median filtering and filtering in the frequency domain.
- Developing and implementing a robust and fast three-dimensional phase unwrapping algorithm. Also, the developed algorithm will be compared to existing algorithms.
- Testing both developed algorithms on real dynamic objects.

1.5 Contributions

The contributions of this research can be summarized as following:

1. Development and implementation of a three-dimensional Fourier fringe analysis system.
2. Design and development of a three-dimensional phase unwrapping algorithm that searches for the optimal unwrapping path in the three-dimensional phase volume.
3. Design and development of a robust three-dimensional phase unwrapping algorithm, which avoids singularity loops in the phase volume following the optimal unwrapping path.
4. Extension of the existing two-dimensional quality-maps into three dimensions.
5. Investigation into the effect of quality maps on determining the optimal unwrapping path.
6. Comparison of different two-dimensional and three-dimensional phase unwrapping algorithms.
7. Evaluation of the performance of the three-dimensional Fourier fringe analysis and three-dimensional phase unwrapping algorithms on real patients.

1.6 Thesis Structure

The thesis is divided into six chapters as follows:

Chapter 1: Introduction
This introductory chapter outlines non-contact measurement and its applications. Details of the scope of this research are also given. The chapter also describes the structure of the thesis.

Chapter 2: Fringe Pattern Analysis Techniques
This chapter gives a review of a number of techniques used to demodulate fringe patterns, such as Fourier fringe analysis, phase stepping, direct phase detection, and wavelet fringe analysis.

Chapter 3: A Review of Two-Dimensional Phase Unwrapping Algorithms
In this chapter, the concept of phase unwrapping is explained and a detailed review of the existing two-dimensional phase unwrapping algorithms is given.

Chapter 4: Three-Dimensional Fourier fringe analysis
The implementation of a fully three-dimensional form of the Fourier fringe analysis technique is the main subject in this chapter. Also, this chapter shows the results of applying three-dimensional Fourier fringe analysis on simulated as well as real dynamic objects.

Chapter 5: Three-Dimensional Phase Unwrapping Algorithms
In this chapter, a review of existing three-dimensional phase unwrapping algorithms is presented. Moreover, this chapter proposes two novel three-dimensional phase unwrapping algorithms that utilise quality maps to find an optimal unwrapping path.
The proposed algorithms are applied here to the unwrapping of both computer simulated and real phase volumes. Furthermore, this chapter compares the proposed techniques with other existing three-dimensional phase unwrapping algorithms.

Chapter 6: Conclusions and Future Work
This is the final chapter in the thesis, where conclusions are drawn from the research described in this thesis. Also, a list of possible future work is outlined in this chapter.
References:


Chapter Two

Fringe Pattern Analysis Techniques
Chapter Two
Fringe Pattern Analysis Techniques

2.1 Introduction

Many techniques have been proposed for the analysis of fringe patterns. These techniques vary in accuracy, the number of frames required and processing time. The aim of any fringe pattern analysis algorithm is to obtain the phase information encoded into the fringe pattern. This phase may be wrapped between \((-\pi, \pi)\) and needs to be unwrapped, as will be shown in the next chapter.

Fringe pattern analysis algorithms can be classified into two categories: spatial and temporal techniques. Spatial algorithms calculate the phase of a pixel in a fringe pattern depending on its neighbouring pixels. Examples of a spatial technique are; Fourier fringe analysis, wavelet transform fringe analysis and direct phase demodulation. Spatial techniques require at least one fringe pattern to calculate phase components. Conversely, temporal algorithms require at least three images to calculate the phase components of a fringe pattern. Temporal techniques calculate the phase of a pixel depending on the values of that pixel in different images and independent of its surrounding pixels. An example of a temporal algorithm is phase stepping. All major previous techniques are explained and reviewed in this chapter.

2.2 Direct Phase Demodulation

Direct phase detection (DPD) adopts phase detection methods from communications theory that are used to demodulate FM radio signals and applies it to analyse fringe patterns (Ichioka and Inuiya, 1972).
A block diagram of a direct phase detector is shown in Figure 2.1 (Hobson et al., 1997). As shown in the figure, the fringe pattern is applied to a Band Pass Filter (BPF) in order to remove the background illumination and the high frequency noise. Then the filtered fringe pattern is split into two parts and each part is applied to a multiplier as shown in Figure 2.1. The output of the multiplier is applied to a Low Pass Filter (LPF) in order to attenuate the high frequency components that result from the multiplication process. An arctangent function is used to extract the desired phase information from the outputs of the low pass filters as indicated in the diagram. The resultant phase map is returned wrapped between (-π, π) and hence an unwrapping algorithm is required in order to remove the 2π ambiguities.

![Figure 2.1: A Diagram of Direct Phase Detection](image)

As stated in the previous chapter, the intensity profile of any fringe pattern is given by:

$$g(x, y) = a(x, y) + b(x, y) \cos(2 \pi f_o x + \phi(x, y))$$

(2.1)

Where $a(x, y)$ represents the background illumination, $b(x, y)$ the amplitude modulation of fringes, $f_o$ the spatial carrier frequency, $\phi(x, y)$ the phase modulation of fringes (the required phase distribution) and $x$ & $y$ the sample indices for the $x$ and $y$ axes respectively.
Applying this fringe pattern to the BPF will remove the low frequency component $a(x, y)$. The filtered image is given by:

$$g_{BPF}(x, y) = b(x, y) \cos(2\pi f_o x + \varphi(x, y))$$  \hfill (2.2)

This filtered image will be applied to the multipliers as shown in Figure 2.1 and the output images are:

$$I_1(x, y) = b(x, y) \cos(2\pi f_o x + \varphi(x, y)) \cos(2\pi f_o x)$$  \hfill (2.3)

$$Q_1(x, y) = -b(x, y) \cos(2\pi f_o x + \varphi(x, y)) \sin(2\pi f_o x)$$  \hfill (2.4)

Applying trigonometric identities in the above equation will yield:

$$I_1(x, y) = \frac{b(x, y)}{2} \left[ \cos(4\pi f_o x + \varphi(x, y)) + \cos(\varphi(x, y)) \right]$$  \hfill (2.5)

$$Q_1(x, y) = -\frac{b(x, y)}{2} \left[ \sin(4\pi f_o x + \varphi(x, y)) - \sin(\varphi(x, y)) \right]$$  \hfill (2.6)

Applying these two images to the LPF will remove the high frequency components and the filtered images are given by:

$$I(x, y) = \frac{b(x, y)}{2} \cos(\varphi(x, y))$$  \hfill (2.7)

$$Q(x, y) = \frac{b(x, y)}{2} \sin(\varphi(x, y))$$  \hfill (2.8)

The phase distribution can be calculated by:

$$\psi(x, y) = \tan^{-1} \left( \frac{I(x, y)}{Q(x, y)} \right) = \varphi(x, y) \mod \pi$$  \hfill (2.9)

This phase distribution is wrapped between (-\pi, \pi), and needs to be unwrapped for future processing.
This direct phase demodulation method is applied to analyse a simulated fringe pattern shown in Figure 2.2. This fringe pattern has been produced by using a computer simulated object shown in Figure 2.3. This simulated fringe pattern can be described by:

\[
g_{\text{sim}}(x, y) = noise_1(x, y) + \left[1 + noise_2(x, y)\right] \cos \left(2\pi f^* x + 10^* \left(\frac{\sin(x)}{x} + \frac{\sin(y)}{y}\right)\right) \tag{2.10}
\]

Where,

- \(noise_1(x, y)\) is a Gaussian noise with zero mean and standard deviation of 0.25.
- \(noise_2(x, y)\) is a Gaussian noise with zero mean and standard deviation of 0.05.

Note that these values of noise are chosen arbitrary.

Figure 2.2: A Computer simulated fringe pattern
Figure 2.3: A Computer simulated object used to produce the fringe pattern shown in Figure 2.2

This computer simulated fringe pattern is applied to the BPF then to the multipliers as described earlier. Figure 2.4 shows the output of each multiplier denoted as $I_i(x,y)$ and $Q_i(x,y)$ respectively in Figure 2.1. The images shown in Figure 2.4 are filtered using LPFs and the filtered images are shown in Figure 2.5. The wrapped phase image is shown in Figure 2.6.

Figure 2.4: The output of the multipliers shown in Figure 2.1 (a) Image $I_i(x,y)$ yielded from the upper multiplier; (b) Image $Q_i(x,y)$ yielded from the lower multiplier.
The wrapped phase distribution shown in Figure 2.6 needs to be unwrapped in order to be usable by any future processes, i.e., to be used in calculating the object’s height. Figure 2.7 shows a three-dimensional view of the unwrapped phase distribution yielded by applying the two-dimensional phase unwrapping algorithm proposed by Herraez et al. (Herraez et al., 2002). A detailed review of two-dimensional phase unwrapping algorithms will be covered in the next chapter.
Figure 2.7: A three-dimensional view of the unwrapped phase which was yielded by unwrapping the wrapped phase map shown in Figure 2.5.

2.3 Phase Stepping

Phase stepping is a very robust technique for the analysis of fringe patterns. It produces a wrapped phase distribution by using at least three source fringe patterns. In comparison to single frame measurement techniques, such as Fourier fringe analysis, phase stepping is computationally relatively simple.

Phase stepping methods have been used since 1974 (Bruning and al., 1974) and have been developed by many researchers (Hariharan et al., 1987; Wyant and Creath, 1992; Creath, 1993; Creath and Schmit, 1994; Hibino et al., 1995). Many different variants of the phase stepping algorithms have been proposed, such as the three-frame, Carré, four-frame, five-frame and the "2+1" techniques (Creath, 1993).
As explained above, the intensity of a fringe pattern can be expressed as:

\[ g(x, y) = a(x, y) + b(x, y) \cos(2\pi f_o x + \phi(x, y)) \]

Three unknowns exist in this equation: \( a(x,y) \), \( b(x,y) \) and \( I(x,y) \). The phase \( \phi(x,y) \) is the information of interest and the other two terms \( a(x,y) \) and \( b(x,y) \) need to be eliminated. To determine the required phase information, at least three independent equations are required.

A common phase stepping technique is the four-frame technique, which uses four fringe patterns with a deliberately inserted, accurately-known phase shift of \( \pi/2 \) radians between them. The intensity in the four fringe patterns can be expressed as:

\[
\begin{align*}
g_1(x, y) &= a(x, y) + b(x, y) \cos(2\pi f_o x + \phi(x, y)) \\
g_2(x, y) &= a(x, y) + b(x, y) \cos(2\pi f_o x + \phi(x, y) + \frac{\pi}{2}) \\
g_3(x, y) &= a(x, y) + b(x, y) \cos(2\pi f_o x + \phi(x, y) + \pi) \\
g_4(x, y) &= a(x, y) + b(x, y) \cos(2\pi f_o x + \phi(x, y) + \frac{3\pi}{2})
\end{align*}
\]

(2.11)

Subtracting \( g_3(x,y) \) from \( g_1(x,y) \) yields:

\[
g_1(x, y) - g_3(x, y) = 2b(x, y) \cos(2\pi f_o x + \phi(x, y))
\]

(2.12)

Also subtracting \( g_2(x,y) \) from \( g_4(x,y) \) gives:

\[
g_4(x, y) - g_2(x, y) = 2b(x, y) \sin(2\pi f_o x + \phi(x, y))
\]

(2.13)

Dividing the last two equations and taking the arctangent of the result gives the desired phase information:

\[
\psi(x, y) = \tan^{-1} \left( \frac{g_4(x, y) - g_2(x, y)}{g_1(x, y) - g_3(x, y)} \right) = \left[ 2\pi f_o x + \phi(x, y) \right] \mod \pi
\]

(2.14)
The resultant phase map is wrapped and an unwrapping algorithm is required in order to remove the $2\pi$ steps.

The four-frame phase stepping algorithm is implemented on a computer-generated fringe pattern shown in Figure 2.2. Figure 2.8 shows four different fringe patterns with 0, $\pi/2$, $\pi$ and $3\pi/2$ radian phase shifts in (a), (b), (c) and (d) respectively.

Figure 2.9 shows the results of the subtraction operation expressed in Equations 2.12 and 2.13. As shown in the figure, the background noise $a(x,y)$ is completely suppressed in this stage due to the fact that similar noise distributions exist in each frame and are cancelled by the subtractions.

Figure 2.8: Phase stepping images showing the four steps of the simulated fringe pattern shown in Figure 2.2. The fringes patterns with (a) 0, (b) $\pi/2$, (c) $\pi$ and (d) $3\pi/2$ phase shifts.
Three-Dimensional Fourier Fringe Analysis and Phase Unwrapping

Chapter 2

Figure 2.9: The results of subtracted images. (a) Result of subtracting the frame with $\pi$ phase shift from the 0 phase shifted frame, (b) result of subtracting the frame with $\pi/2$ phase shift from the $3\pi/2$ phase shifted frame.

The wrapped-phase map resulted from applying the four-frame phase stepping algorithm is shown in Figure 2.10.

Figure 2.10: The wrapped-phase map resulting from the use of the four-frame phase stepping algorithm on a computer generated object.
Figure 2.11 shows a 3-D surface height of the unwrapped phase map. Figure 2.11(a) shows the unwrapped phase map without removing the carrier frequency, whereas Figure 2.11(b) shows the unwrapped and tilt-removed phase map (i.e. with the carrier frequency removed).

Figure 2.11: Three-dimensional view of the unwrapped phase resulting from unwrapping the wrapped phase shown in Figure 2.9. (a) Without tilt removal & (b) with tilt removal
2.4 Fourier Transform Fringe Analysis.

Fourier Transform Fringe Analysis (FTFA), also known as Fourier Fringe Analysis (FFA) or Fourier transform Profilometry (FTP), is a very popular technique that is well known to people working in the field of non-contact measurement. Since Takeda proposed his technique for using the Fourier transform in analysing a fringe pattern in 1982 (Takeda et al., 1982), many applications have adopted this technique in order to measure 3D surface shape. A lot of further research has been carried out to improve the performance and quality of this technique.

As mentioned earlier, fringe analysis techniques have to extract the phase signal $\phi(x, y)$ from the intensity distribution of the source fringe pattern. An intensity profile $g(x, y)$ in a fringe pattern may be given by the following equation:

$$g(x, y) = a(x, y) + b(x, y) \cos(2\pi f_o x + \phi(x, y))$$

In 1982, Takeda et al. suggested the use of a one-dimensional Fourier transform as a powerful tool to extract the phase information encoded in a fringe pattern (Takeda et al., 1982).

Takeda expressed the cosine term in Equation (2.1) as a summation of two exponents as shown in the following equation:

$$g(x, y) = a(x, y) + c(x, y)e^{i2\pi f_o x} + c^*(x, y)e^{-i2\pi f_o x}$$  \hspace{1cm} (2.15)

where,

$$c(x, y) = \frac{b(x, y)e^{i\phi(x, y)}}{2}$$  \hspace{1cm} (2.16)

$$c^*(x, y) = \frac{b(x, y)e^{-i\phi(x, y)}}{2}$$  \hspace{1cm} (2.17)
Taking the one-dimensional Fourier Transform we will have:

\[ G(f_x, y) = A(f_x, y) + C(f_x - f_0, y) + C^*(f_x + f_0, y) \] \hspace{1cm} (2.18)

where, \( A(f_x, y) \) describes the spectrum of the background illumination, \( C(f_x - f_0, y) \) and \( C^*(f_x + f_0, y) \) are the spectra of the deformed fringes. By selecting either the \( C \) or \( C^* \) term using the window \( H \) as shown in Figure 2.12 below and shifting it toward the origin, as shown in Figure 2.13, we will isolate the term \( C(f_x, y) \) which contains the desired information.

![Figure 2.12: The frequency spectrum of fringe pattern.](image1)

![Figure 2.13: The desired frequency component](image2)
By transforming the term \( C(f_x, y) \) back into its original spatial domain using the one-dimensional Inverse Fourier Transform (1D-IFT), we will get \( c(x, y) \).

The real and imaginary parts of \( c(x, y) \) are given by:

\[
\text{Re}[c(x, y)] = b(x, y) \cos(\varphi(x, y)) \tag{2.19}
\]
\[
\text{Im}[c(x, y)] = b(x, y) \sin(\varphi(x, y)) \tag{2.20}
\]

And the phase information is given by:

\[
\psi(x, y) = \tan^{-1} \left( \frac{\text{Im}[c(x, y)]}{\text{Re}[c(x, y)]} \right) = \varphi(x, y) \mod \pi \tag{2.21}
\]

Because of the use of a trigonometric arctangent function, the extracted phase will be in the range \(-\pi\) and \(\pi\), and will need to be unwrapped using phase unwrapping algorithms, which will be explained in the next chapter.

One-dimensional FFA is now applied to analyse the simulated fringe pattern shown in Figure 2.2, on a row by row basis. Where each row is analysed as described earlier. Figure 2.14(a) shows the resultant wrapped-phase map, whereas the unwrapped-phase surface is shown in Figure 2.14(b). Apparently from the figure, the 1D-FFA technique is less robust in analysing the example fringe pattern when compared to the phase-stepping and direct phase detection methods.

Since 1982, FFA has witnessed many improvements. Bone et al. introduced one major improvement in 1986 (Bone et al., 1986). Bone proposed the use of two-dimensional Fourier transform to analyse the fringe pattern as a two-dimensional image. The two-dimensional transform permits better separation of the desired information components from the unwanted components. Using two dimensional FFA also increased the quality.
of the results, with a corresponding slight increase in the processing time (Bone et al., 1986).

Figure 2.14: (a) The wrapped-phase map resulting from using 1D-FFA. (b) A 3D representation of the unwrapped-phase resulting from using one-dimensional FFA.

If the 2D-FFA technique is now applied to analyse the simulated fringe pattern shown in Figure 2.2, the results are as shown in Figure 2.14. This figure shows the individual
stages of the two-dimensional FFA method. Figure 2.15(a) shows the frequency spectrum of the fringe pattern. The desired component is selected using a two-dimensional exponential filter, which is shown in Figure 2.15(b), and the resulting filtered component is shown in Figure 2.15(c). Figure 2.16(a) shows the resultant wrapped-phase map in 2D form, whereas the unwrapped-phase surface is shown in Figure 2.16(b) as a 3D isometric plot. Comparing Figures 2.14(b) and 2.16(b) shows the great benefit in accuracy and data quality that is obtained from moving from 1D-FFA to 2D-FFA. It is this type of improvement that is a prime motivation towards investigating the potential benefits that could be obtained by taking into account a further third dimension in such analysis techniques.

Figure 2.15: The use of two-dimensional FFA. (a) the frequency spectrum of the fringe-pattern shown in Figure 2.13(a), (b) 2-D filter to select the desired frequency component. (c) The selected frequency component.
Figure 2.16: (a) The wrapped-phase map that resulted from using 2D-FFA. (b) A 3D representation of the unwrapped-phase that resulted from using 2D-FFA.
After the two-dimensional Fourier fringe analysis algorithm was first published, it was subject to intensive study in order to improve its performance. Many researchers proposed an elimination of the zero frequency component, in order to make the frequency selection much easier and to extend the measurable slope of the height variation. For example; Li et al. in 1990 (Li et al., 1993) proposed a method to eliminate the zero frequency component by projecting two fringe patterns onto the object with a phase shift of $\pi$ radians between them, and hence subtracting the patterns from each other effectively eliminates the zero frequency component.

The Fourier fringe analysis algorithm suffers from some problems however, such as leakage and difficulty in resolving sharp-edges. Frequency leakage can be minimized by multiplying the fringe pattern by a smoothing window, such as a Hamming window. Fiona Berryman et al. in 2004, studied the effect of windowing in the Fourier transform, and showed the performance of applying different types of windows (Berryman et al., 2004). A challenging problem is to distinguish between a discontinuity produced by a real sharp edge on the object and that produced by a phase discontinuity. David Burton et al. (Burton and Lalor, 1994) proposed a technique termed as multi-channel Fourier fringe analysis to aid phase unwrapping algorithms in the presence of surface discontinuities, which projected different patterns at different angles and frequencies to resolve the surface discontinuities. The basic idea of this algorithm is that the surface discontinuities will appear on all fringe patterns. David Burton et al. in 1995 (Burton et al., 1995) show the effect of removing the carrier frequency by carrier frequency shifting in order to minimize the complexity of the wrapped phase map and to avoid phase discontinuities in some particular applications. Recently Chen et al. in 2005 (Chen et al., 2005) proposed an elimination of the zero spectrum using a windowed Fourier transform.

Nowadays, the concern is increasingly towards measuring dynamic objects. Xianyu Su et al. in 2001 (Su et al., 2001) proposed the use of two-dimensional FFA for shape measurement of dynamic objects. In this method a CCD camera rapidly grabs a sequence of images. These sequences are analysed frame by frame using 2D-FFA to produce a wrapped phase volume which can be unwrapped using 2D or 3D phase unwrapping algorithms, as will be shown later. Xianyu Su et al. in 2001 (Su and Chen, 2001) presented a thorough review about Fourier transform fringe pattern analysis.
2.5 Wavelet Transform Fringe Analysis.

Recently, there has been much interest in phase demodulation of fringe patterns using wavelet transforms. Wavelet transform fringe analysis is a frequency-domain fringe analysis technique, in which the demodulation of the phase is carried out by transferring the fringe pattern into a different domain. It is well known in digital signal processing theory that the wavelet transform technique is more suitable for the analysis of non-stationary signals rather than stationary signals. A stationary signal is a signal whose frequency contents does not change in time, whereas a non-stationary signal is a signal whose frequency contents do change in time (Malat, 1999). Fringe patterns tend to resemble non-stationary signals. This has motivated researchers to investigate wavelet transform techniques for the application of fringe pattern demodulation.

Many researches have proposed one-dimensional continuous wavelet transform (1D-CWT) techniques to demodulate fringe patterns. These algorithms extract the phase of fringe patterns and can be classified into two different approaches: phase estimation (Zhong and Weng, 2004) and frequency estimation (Sandoz, 1997; Cherbuliez et al., 1999; Dursun et al., 2004).

In the phase estimation method, the fringe pattern is applied to the 1D-CWT algorithm on a row by row basis and the resultant wavelet transform is a two-dimensional complex array. To extract the phase, this complex array is decomposed into two arrays, one for the absolute values and the other for the phase values. Then, from the absolute value array, the maximum value of each column in that array has to be extracted using one of several available ridge extraction algorithms (Delprat et al., 1992; Carmona et al., 1997; Liu et al., 2004). After extracting the ridge, the wrapped phase of the row under processing is represented by the corresponding phase values of that ridge. Then the wrapped phase has to be unwrapped, after which it can be used to calculate the actual height values. A review of phase unwrapping algorithms will be given in the next chapter. This process is repeated until all lines in the fringe pattern are analysed.

The second approach for retrieving the phase of a fringe pattern is to estimate the instantaneous frequencies in the fringe pattern. In the frequency estimation method, the
ridge of maximum values has to be extracted in a similar manner to that used in the phase estimation method. After identifying the ridge, the scale values are extracted from the positions where this ridge is located. The unwrapped phase is then calculated by integrating the scale values. In this method phase unwrapping is not required, as the integration of the scale values will directly produce the unwrapped phase. Gdeisat et. al. show that the frequency estimation method is not reliable and it fails to demodulate real fringe patterns (Gdeisat et al., 2006).

Figure 2.17 shows an illustration of the use of the phase estimation method to extract the phase from the simulated fringe pattern that was originally shown in Figure 2.2. The implementation of the wavelet fringe analysis algorithm has been carried out by author’s colleague working on wavelet fringe analysis area in Liverpool John Moores University. Figure 2.17(a) shows the absolute value array resulting from applying a row of this fringe pattern to a 1D-CWT. As shown in Figure 2.17(a), the ridge of maximum values is extracted and is represented in this figure by a solid line. Figure 2.17(b) shows the phase value array for the same row. The phase values correspond to the ridge extracted in Figure 2.17(a) and represent the wrapped phase for that particular row. Figure 2.17(c) shows the complete wrapped phase map resulting from analysing all rows and a 3D view of the unwrapped phase is shown in Figure 2.17(d).

Recently, many researchers have proposed the use of the 2D-CWT to analyse fringe patterns (Kadooka et al., 2003; Gdeisat et al., 2006). Using the 2D-CWT is more robust for demodulation of fringe patterns and has better noise immunity than the 1D-CWT method (Gdeisat et al., 2006).
Figure 2.17: Wavelet transform fringe analysis. (a) the extracted ridge from the modulus array resulting from applying one row of fringe pattern shown in Figure 2.2 to the 1D-CWT, (b) the corresponding ridge in the phase array, (c) the wrapped phase map and (d) the unwrapped phase map

(Results obtained from Mr. Abdulbasit Abid, Liverpool John Moores University)
2.6 Summary

There are many algorithms that can be used to analyse fringe patterns. A number of these algorithms have been reviewed in this chapter. Phase stepping techniques require low computational complexity. The main drawback of the phase stepping technique is the requirement for more than one frame to be captured (three at least) for the analysis of an object. This requirement makes phase stepping an inappropriate choice for dynamically moving objects. FFA and WTFA offer good results and only require a single frame to be captured in order to produce a result. On the other hand, FFA and WTFA are very complicated algorithms. Generally speaking, there is no ideal fringe analysis technique that can withstand the very wide variety of different practical requirements. Every technique has advantages and drawbacks.

Table 2.1 shows a comparison between the four algorithms discussed in this chapter.

<table>
<thead>
<tr>
<th>Analysis domain</th>
<th>DPD¹</th>
<th>Phase stepping</th>
<th>FFA²</th>
<th>WFA³</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computational complexity</td>
<td>Medium</td>
<td>Low</td>
<td>High</td>
<td>Very High</td>
</tr>
<tr>
<td>Noise Robustness</td>
<td>Medium</td>
<td>Medium</td>
<td>High</td>
<td>High</td>
</tr>
<tr>
<td>Number of frames required</td>
<td>One frame</td>
<td>Three frames at least</td>
<td>One frame</td>
<td>One frame</td>
</tr>
<tr>
<td>Possibility for analysing dynamic objects</td>
<td>Possible</td>
<td>Not possible</td>
<td>Possible</td>
<td>Possible</td>
</tr>
<tr>
<td>Coping with discontinuous and high detailed objects</td>
<td>Not good</td>
<td>Very good</td>
<td>Not good</td>
<td>Not good</td>
</tr>
<tr>
<td>Ease of automation</td>
<td>Low</td>
<td>High</td>
<td>Low</td>
<td>Medium</td>
</tr>
</tbody>
</table>

¹ DPD: Direct Phase Detection.
² FFA: Fourier Fringe Analysis.
³ WFA: Wavelet Fringe Analysis.
References:


Three-Dimensional Fourier Fringe Analysis and Phase Unwrapping  

Chapter 2


Chapter Three

A Review of 2D Phase Unwrapping Algorithms
Chapter Three

A Review of Two-Dimensional Phase Unwrapping Algorithms

3.1 Introduction

Many fringe analysis techniques such as; Fourier transform fringe analysis, phase stepping, wavelet transform fringe analysis… etc, end up with the use of the arctangent function, as shown in the previous chapter. Due to the use of this trigonometric function, the extracted phase is typically wrapped into the interval \((-\pi, \pi)\) and discontinuities of values of \(2\pi\) appear and thus need to be unwrapped in order to provide the required continuous phase information.

Phase unwrapping can be defined to be “the process by which the absolute value of the phase angle of a continuous function that extends over a range of more that \(2\pi\) is recovered. This absolute value is lost when the phase term is wrapped upon itself with a repeat distance of \(2\pi\) due to the fundamental sinusoidal nature of the wave functions used in measurement of physical properties” (Robinson and Reid, 1993). Or simply it can be also be defined as “the process of solving the ambiguity problem caused by the fact that the absolute phase is wrapped into the interval \((-\pi, \pi)\), recovering the continuous phase information from the wrapped phase” (Gens, 2003).

In the case where there are no disturbances in the phase data, the unwrapped phase can be easily obtained by integrating the phase gradients over the whole data set and this may be performed in a manner that is completely independent of the actual integration path. But, there are several sources of disturbances that cause this simple unwrapping approach to fail in the vast majority of cases. For example: under-sampling, noise, object discontinuities or fringe breaks, holes or missing phase information, … etc (Huntley, 1989).
During the last three decades, the field of two-dimensional phase unwrapping has been intensively studied and hundreds of journal papers have been published. Many techniques were proposed to solve the phase unwrapping problem. These algorithms vary in accuracy and computational requirements. In most cases, more accurate results require more complex computations. These techniques can be roughly classified into two major categories; local and global phase unwrapping algorithms. A complete review of the two-dimensional phase unwrapping problem has been presented by Ghiglia and Pritt (Ghiglia and Pritt, 1998).

Local phase-unwrapping algorithms, also called path-following algorithms, unwrap the phase map by integrating the phase gradients over a certain path that connects all pixels in the wrapped phase map. Unwrapped phase depends on the choice of the integration path, i.e. if the unwrapping algorithm follows two different paths from one pixel to another, it may produce two different answers for the unwrapped phase.

Global phase-unwrapping algorithms formulate the unwrapping process in terms of minimization of a global function to estimate the phase gradient; they do not rely on an integration path to perform unwrapping. All the algorithms in this class are known to be robust but computationally intensive. A comparison between local and global phase unwrapping algorithms was carried out by Fornaro et al. (Fornaro et al., 1997a; Fornaro et al., 1997b).

In this chapter, the major local and global phase unwrapping algorithms will be reviewed. In section 3.2, local phase unwrapping algorithms are discussed and explained. Global algorithms are discussed in section 3.3.

3.2 Local phase unwrapping algorithms

Local phase unwrapping algorithms find the unwrapped phase values by integrating the phase along a certain path. They follow an integration path that covers the whole wrapped phase map and this is why local phase unwrapping algorithms are also called path-following algorithms (Ghiglia and Pritt, 1998). In the local phase unwrapping methods the unwrapped phase is defined by:
\[ \varphi(b) = \varphi(a) + \sum_{i=a}^{b-1} \Delta(\varphi(i)) \]  

(3.1)

where;

- \( \varphi(b) \) the unwrapped phase at an arbitrary point \( b \)
- \( \varphi(a) \) the unwrapped phase at the initial point \( a \)
- \( \Delta \) is the difference operator where

\[ \Delta\{\varphi(n)\} = \varphi(n+1) - \varphi(n) \]  

(3.2)

Equation 3.1 states that the unwrapped phase value at a pixel \( b \) is equal to the unwrapped phase value at the pixel \( a \) plus the summation (integration) of the unwrapped phase differences among a certain path from \( a \) to \( b \). Actually, Equation 3.1 is a general equation that can be applied to any continuous function, which states that any function can be expressed as a summation of its difference. On that basis, Equation 3.1 cannot be used to determine the unwrapped phase solution and needs to be modified.

The unwrapped phase values and the wrapped phase values can be related with each other by: (Itoh, 1982)

\[ \psi(n) = \varphi(n) + 2\pi \cdot k(n) \]  

(3.3)

\[ -\pi < \psi(n) \leq \pi \]

\[ \varphi(n) = \psi(n) + 2\pi \cdot v(n) \]  

(3.4)

\[ -\infty < \varphi(n) < \infty \]

where;

- \( \psi(n) \) is the wrapped phase values.
- \( \varphi(n) \) is the unwrapped phase values.

k(n) is the function containing the integers that must be added to the unwrapped phase \( \phi \) to be wrapped between \((-\pi, \pi)\).

\( n \) is an integer.

\( \nu(n) \) is the function containing a set of integers that must be added to the wrapped phase \( \psi \) to be unwrapped.

Noting that:

\[ \nu(n) = -k(n) \]  
(3.5)

Define the wrapping operation \( w \) to be the operation which converts the unwrapped phase \( \phi(n) \) to the wrapped phase, or mathematically:

\[ w\{\phi(n)\} = \arctan\left(\frac{\sin(\phi(n))}{\cos(\phi(n))}\right) \]  
(3.6)

or;

\[ w\{\phi(n)\} = \psi(n) = \phi(n) + 2\pi \cdot k(n) \]  
(3.7)

The difference in the wrapped phase is given by:

\[ \Delta\{\psi(n)\} = \psi(n + 1) - \psi(n) \]  
(3.8)

Substitute the value of the wrapped phase in Equation 3.3 into Equation 3.8 yield:

\[ \Delta\{\psi(n)\} = \Delta\phi(n) + 2\pi \cdot \Delta k(n) \]  
(3.9)

Applying the wrapping operator on the wrapped phase difference \( \Delta\{\psi(n)\} \) we will get:
\[
\begin{align*}
\Delta \psi(n) &= \arctan \left[ \frac{\sin(\Delta \psi(n))}{\cos(\Delta \psi(n))} \right] \\
&= \arctan \left[ \frac{\sin(\Delta \varphi(n) + 2\pi \Delta k(n))}{\cos(\Delta \varphi(n) + 2\pi \Delta k(n))} \right] \\
&= \arctan \left[ \frac{\sin(\Delta \varphi(n))}{\cos(\Delta \varphi(n))} \right]
\end{align*}
\]

so:

\[
w[\Delta \psi(n)] = w[\Delta \varphi(n)]
\]

When there is no phase aliasing:

\[
\text{if } |\Delta \varphi(n)| < \pi, \text{ then } w[\Delta \varphi(n)] = \Delta \varphi(n)
\]

so:

\[
w[\Delta \psi(n)] = \Delta \varphi(n)
\]

Substitute equation 3.7 into equation 3.1 we will get:

\[
\varphi(b) = \varphi(a) + \sum_{i=a}^{b-1} w[\Delta \psi(i)]
\]

Where:

\(\varphi(b)\) is the unwrapped phase at the point \(b\).

\(\varphi(a)\) is the unwrapped phase at an arbitrary starting point \(a\), noting that at the initial point the unwrapped phase is assumed to be equal to the wrapped phase.

Equation 3.14 states that in the local phase unwrapping algorithms the unwrapping is “an integration of wrapped phase differences along the unwrapping path”.

In reality, unwrapping the wrapped phase using Equation 3.14 will propagate errors through the unwrapped phase map. Errors from unreliable regions propagate to reliable regions and cause a complete failure in the unwrapped-phase map. So, completely
relying on equation 3.14 may cause the propagation of errors throughout the whole unwrapped phase.

Many local phase unwrapping algorithms have been proposed to prevent error propagation in the unwrapped phase. These algorithms attempt to isolate unreliable regions in the phase map and minimize their effects on the unwrapped result. Local phase unwrapping methods can be classified into two main types. The first type defines the quality of each pixel in the phase map in order to unwrap the highest quality pixels first and the lowest quality pixels last. These algorithms are called quality-guided phase unwrapping algorithms. The second type is known as residue-balancing methods. Residue-balancing methods attempt to prevent error propagation by identifying residues (the source of noise in the wrapped phase). These residues must be balanced and isolated by using barriers (branch-cuts). Once these branch cuts are identified the unwrapping path should not pass through them. This prevents error propagation. Both types of algorithm will be explained in detail in the following subsections.

3.2.1 Quality-Guided Phase Unwrapping Algorithms.

Quality-guided algorithms depend on a quality measure to guide the unwrapping path. The main idea of these algorithms is to unwrap the highest quality pixels first and the lowest quality pixels last to prevent error propagation (Herraez et al., 1996; Xu and Cumming, 1996). The success or failure of these algorithms depends strongly upon the production of a good quality map. The quality map can be defined as an array of values that define the quality or goodness of each pixel of the given phase map. Several two-dimensional quality-guided algorithms have been proposed over the last few decades. Bone (Bone, 1991) was the first who adopted a quality measure to guide the unwrapping process. He calculated the second-order partial derivative of each phase data point and used it to determine its quality. The matrix of quality values is called a quality map. Only those pixels whose qualities exceed a certain threshold are unwrapped, others are not processed until they exceed the threshold value, which is reduced with each subsequent iteration.
Subsequently, many quality-guided, path following algorithms were proposed (Quiroga and Bernabeu, 1994; Herraez et al., 1996; Xu and Cumming, 1996; Herraez et al., 2002b; Herraez et al., 2005). They design different unwrapping paths that rely completely upon the quality map to guide the integration path. They essentially unwrap the highest quality pixels first and leave the lowest quality pixels until last.

Herraez et al. were the first to introduce the concept of using the quality of edges to guide the unwrapping path. An edge can be defined as the connection between adjacent pixels, as shown in Figure 3.1. Herraez et al. proposed a discrete unwrapping path based on using the qualities of the edges (Herraez et al., 2002a). This algorithm is very robust and fast and has been used in constructing a robust fringe pattern analysis system for human body shape measurement (Lilley et al., 2000).

A drawback of these algorithms is that there is no guarantee that the path will not encircle an unbalanced residue and introduce a spurious discontinuity in the unwrapped solution.

3.2.2 Two-Dimensional Quality Maps

A very important feature of any quality-guided phase unwrapping algorithm is the utilisation of a reliable quality map. A good quality map determines the success or failure of the phase unwrapping algorithms. Quality maps provide additional information about the wrapped phase map. This additional information provides a measure of the quality of a particular pixel within the wrapped phase map.
Quality maps can be calculated using different methods (Ghiglia and Pritt, 1998). The most commonly used methods for calculating quality-maps in the phase unwrapping field are the second difference, phase derivative variance, maximum phase gradients and pseudo-correlation techniques.

### 3.2.2.1 Two-Dimensional Pseudo-Correlation Quality Map

The pseudo-correlation method attempts to mimic a correlation quality map which only exists for certain applications, such as synthetic aperture radar. It estimates the quality map from the phase data itself, without the need for an extra set of data.

The pseudo-correlation quality map defines the goodness of each pixel using the Equation:

\[
q(m,n) = \sqrt{\left(\sum \cos(\psi(i,j))\right)^2 + \left(\sin(\psi(i,j))\right)^2} / k^2
\]  

(3.15)

where

\(\psi(i,j)\) is the wrapped phase value of the pixel \((m, n)\),

\(i,j\) represent the neighbours’ indices for the pixel \((m,n)\) in a \(k\times k\) window.

\(k\) is the window size which should be an odd number.

### 3.2.2.2 Two-Dimensional Phase Derivative Variance Quality Map

The phase derivative variance method measures the statistical local variance of the wrapped phase derivatives. It is probably the most reliable quality map that can be extracted from the wrapped phase itself (Ghiglia and Pritt, 1998). Actually, phase derivative variance indicates the badness rather than the goodness of the phase data.

The calculation of the phase derivative variance for a pixel \((m, n)\) using a \(k\times k\) window is defined by the equation:

\[
PV(m,n) = \frac{1}{k^2} \left( \sqrt{\sum \left(\Delta^x\psi(i,j) - \Delta^x\psi(i,j)\right)^2} + \sqrt{\sum \left(\Delta^y\psi(i,j) - \Delta^y\psi(i,j)\right)^2} \right)
\]

(3.16)
where;
\[ \tilde{\Delta}'\psi(i,j) \] and \[ \tilde{\Delta}'\psi(i,j) \] are the wrapped phase gradients in the \( x \) and \( y \) directions respectively.
\[ \bar{\Delta}'\psi(i,j) \] and \[ \bar{\Delta}'\psi(i,j) \] are the mean of the values in a \( k \times k \) cube in \( \Delta' \) and \( \Delta' \) respectively.
\( i,j \) are the neighbours’ indices for the pixel \((m,n)\) in a \( k \times k \) window.

\[ \tilde{\Delta}'\psi(i,j) \] and \[ \tilde{\Delta}'\psi(i,j) \] are defined by:

\[
\tilde{\Delta}'\psi(i,j) = w\{ \psi(i+1,j) - \psi(i,j) \}
\]
\[
\tilde{\Delta}'\psi(i,j) = w\{ \psi(i,j+1) - \psi(i,j) \}
\] (3.17)

Where \( w \) defines a wrapping operator given previously in Equation 3.6.

Finally, the quality of the pixel is defined to be the reciprocal of the variance as described by the equation;

\[ q(m,n) = \frac{1}{PV(m,n)} \] (3.18)

3.2.2.3 Two-Dimensional Maximum Phase Gradient Quality Map

The maximum phase gradient method measures the magnitude of the largest phase gradient \( i.e., \) partial derivative or wrapped phase difference in a \( k \times k \) window. Similarly to the case for the phase derivative variance quality map, the maximum gradient indicates the badness rather than the goodness of the phase data.

\[ MG(m,n) = \max \left\{ \frac{\tilde{\Delta}'\psi(i,j)}{\bar{\Delta}'\psi(i,j)} \right\} \] (3.19)

3.2.2.4 Two-Dimensional Second Difference Quality Map

Another quality map extraction method is the second difference quality map (Herraez et al., 2002a). In our experience, the second difference quality map is more robust than the
others and it has been found by the author to be the best quality map amongst the others described here. This quality map also measures the badness of each pixel in a \( k \times k \) window, using Equation (3.20):

\[
SD(m, n) = \sqrt{H^2(m, n) + V^2(m, n) + \sum_{n=1}^{k} D^2_{n}(m, n)}
\]  

(3.20)

Where;

\[
H(m, n) = w\{\psi(m-1, n) - \psi(m, n)\} - w\{\psi(m, n) - \psi(m+1, n)\}
\]

\[
V(m, n) = w\{\psi(m, n-1) - \psi(m, n)\} - w\{\psi(m, n) - \psi(m, n+1)\}
\]

\[
D_{1}(m, n) = w\{\psi(m-1, n-1) - \psi(m, n)\} - w\{\psi(m, n) - \psi(m+1, n+1)\}
\]

\[
D_{2}(m, n) = w\{\psi(m+1, n-1) - \psi(m, n)\} - w\{\psi(m, n) - \psi(m-1, n+1)\}
\]

(3.21)

\(H(m, n)\) and \(V(m, n)\) are the horizontal and vertical second differences respectively. \(D_{n}(m, n)\) is the \(n\)th diagonal second difference. The second difference measures the badness of each pixel, so the quality of each pixel is the reciprocal of the second difference value.

### 3.2.3 Residue-Balancing Phase Unwrapping Algorithms.

The second approach in local phase unwrapping methods is known collectively as residue-balancing techniques. These algorithms aim to produce a path-independent wrapped phase map. Path-dependency occurs because of the existence of inconsistent points called residues.

Residue-balancing algorithms search for residues in a wrapped-phase map and attempt to balance positive and negative residues by placing cut lines between them. The reason for these cut lines is to create an unwrapping barrier and prevent the unwrapping path penetrating them.

The residue is identified for each pixel in the phase map by calculating the wrapped gradients in a \(2 \times 2\) closed loop, as shown in Figure 3.2, by the following equation:
\[
\phi = \frac{1}{2\pi} \left( \Re \left[ \frac{\Psi_{i,j} - \Psi_{i+1,j}}{2\pi} \right] + \Re \left[ \frac{\Psi_{i+1,j} - \Psi_{i+1,j+1}}{2\pi} \right] + \Re \left[ \frac{\Psi_{i+1,j+1} - \Psi_{i,j+1}}{2\pi} \right] \right) + \Re \left[ \frac{\Psi_{i,j+1} - \Psi_{i,j}}{2\pi} \right]
\]

(3.22)

where, the operator \( \Re [\ ] \) rounds its argument to the nearest integer.

The result of Equation 3.22 above can only take three possible values; +1, -1 or zero. On that basis, a pixel under test is considered to be a positive residue if the value of \( \phi \) is +1, and it is considered to be a negative residue if the value is -1. On the other hand, the pixel is not a residue if the value of \( \phi \) is zero.

After identifying all residues in the wrapped phase map, these residues have to be balanced by the means of branch cuts. Figure 3.3 shows a branch cut that balances two pairs of opposite-sign residues.

Branch-cuts act as barriers to prevent the unwrapping path going thorough them. If these branch cuts are avoided during the unwrapping process, no errors propagate and the unwrapping path is considered to be path independent. On the other hand, if these branch cuts are penetrated during the unwrapping, then errors propagate throughout the whole phase map, and in this case the unwrapping path is considered to be path dependent.
Figure 3.3: Illustration of branch cuts in the wrapped phase map.

Figure 3.4 illustrates the principle of unwrapping around the branch cuts. In Figure 3.4(a) the unwrapping path has penetrated the branch cut which has been placed so as to balance two residues in the wrapped phase map. These errors propagate and create $2\pi$ discontinuities in the unwrapped phase map. On the other hand, Figure 3.4(b) shows the correct unwrapping path that avoids the branch cut, consequently, error propagation in the unwrapped solution is avoided.
Figure 3.4: Unwrapping path with the existence of the branch cuts, (a) incorrect unwrapping path and (b) correct path.

Several algorithms have been proposed to place branch cuts between residues. These methods can be divided into two types: dipole branch cutting and tree branch cutting. In the dipole branch cutting techniques, the branch cut is only placed either between two residues of opposite sign, or between a residue and a border, as shown in Figure 3.5(a). Whereas, in the tree branch cutting methods, the branch cuts form trees that connect a group of residues. The net charge of any group in a tree has to be zero in order to be balanced. Any unbalanced group is connected to the border. An example of the tree branch cutting technique is shown in Figure 3.5(b).
Huntley was the first person to introduce the dipole branch cutting technique (Huntley, 1989). He used the nearest neighbour algorithm to find the optimal branch cutting distribution in the phase map. Huntley’s method was improved by using more sophisticated search strategies such as: improved nearest neighbour, simulated annealing, minimum-cost matching, stable marriages, reverse simulated annealing and genetic algorithms. These algorithms attempt to find the corresponding dipoles with the minimum total connection length (Buckland et al., 1995; Cusack et al., 1995; Gutmann and Weber, 1999; Karout et al., 2007a).

Recently, Karout et al. proposed a new technique of dipole branch cutting based on what they called a residue vector. They show that the optimal branch cut positioning has to follow the residue vector in order to obtain an accurate result (Karout et al., 2007b).

The tree branch-cut technique was first introduced by Goldstein et al. (Goldstein et al., 1988). In this technique, the pixels are utilized to create the branch cuts, such that any pixel can be marked as a part of branch cut that connects two different residues, as shown in Figure 3.5(b). The major drawback of this algorithm is that the branch cuts may be placed on good pixels, which may cause error propagation. To overcome this problem Flynn introduced the mask-cut algorithm, in which the branch-cut placement relies upon a quality map. In his method, the branch cuts are placed on bad quality pixels to connect residues (Flynn, 1996). Ghiglia and Pritt suggested removing the
dipoles as a pre-processing step in order to enhance the Goldstein algorithm (Ghiglia and Pritt, 1998). The dipoles are removed by connecting them using the nearest neighbour procedure proposed by Huntley (Huntley, 1989).

### 3.2.4 Other Local phase unwrapping algorithms.

A different robust local phase unwrapping algorithm is the minimum discontinuity algorithm. Flynn looked at phase unwrapping from a different point of view; his algorithm finds a solution that actually minimizes the discontinuities (Flynn, 1997). The algorithm utilizes a tree-growing approach that traces paths of discontinuities in the phase, detects the paths that form loops, and adds multiples of $2\pi$ to the phase values enclosed by the loops in order to minimize the discontinuities. It performs this process iteratively until no more loops are detected. The process is guaranteed to converge to a minimum discontinuity solution.

A unique feature of Flynn’s algorithm is that it works with or without a quality map. If a quality map is supplied, then the discontinuities are weighted by the quality values, so that the discontinuities in the low-quality regions do not contribute appreciably to the overall measure of discontinuity. As a result the discontinuities in the resulting unwrapped phase map tend to be confined to the lowest-quality regions.

### 3.3 Global phase unwrapping algorithms

In the previous section, it was stated that local phase unwrapping algorithms follow a certain unwrapping path in order to unwrap the phase. They begin at a grid point and integrate the wrapped phase differences over that path, which ultimately covers the entire phase map. Local phase unwrapping algorithms explicitly (residue-balancing algorithms) or implicitly (quality-guided algorithms) generate branch cuts and define the unwrapping path around these cuts in order to minimize error propagation.

On the other hand, Global phase unwrapping algorithms take a completely different approach to phase unwrapping. These algorithms formulate the phase unwrapping problem in a generalised minimum-norm sense. For that reason, many researchers refer
to the global phase unwrapping algorithms as minimum-norm algorithms (Ghiglia and Pritt, 1998).

Global phase unwrapping algorithms attempt to find the unwrapped phase by minimising a global error function as shown in Equation 3.23:

\[ e^p = \| \text{solution} - \text{problem} \|^p \]  

(3.23)

As shown earlier in this chapter, for a case where there is no undersampling, the gradient of the unwrapped phase is equal to the wrapped gradient of the wrapped phase.

\[ \Delta \phi(x, y) = w \{ \Delta \psi((x, y)) \} \]  

(3.24)

Global phase unwrapping algorithms seek the unwrapped phase whose local gradients in the \( x \) and \( y \) direction match, as closely as possible, the wrapped gradients of the wrapped phase, as described in Equation 3.25:

\[ e^p = \sum_{i=0}^{M-2} \sum_{j=0}^{N-2} \left| \Delta^x \phi(i, j) - \tilde{\Delta} \psi(i, j) \right|^p + \sum_{i=0}^{M-2} \sum_{j=0}^{N-2} \left| \Delta^y \phi(i, j) - \tilde{\Delta} \psi(i, j) \right|^p \]  

(3.25)

where;

\( \Delta^x \phi(i, j) \) and \( \Delta^y \phi(i, j) \) are the unwrapped phase gradients in the \( x \) and \( y \) directions respectively, and they are given by:

\[ \Delta^x \phi(i, j) = \phi(i + 1, j) - \phi(i, j) \]  

(3.26)

\[ \Delta^y \phi(i, j) = \phi(i, j + 1) - \phi(i, j) \]  

(3.27)

\( \tilde{\Delta} \psi(i, j) \) and \( \tilde{\Delta} \psi(i, j) \) are the wrapped values of the wrapped phase gradients in the \( x \) and \( y \) directions respectively, and they are given by:
The wrapping operator $w$ is defined in Equation 3.6 above.

Global phase unwrapping algorithms can be divided into three different categories: unweighted least-squares, weighted least-squares and $L_p$-norm methods. All of these types are discussed in the following subsections.

### 3.3.1 Unweighted Least-squares

The unweighted least squares methods, which was first introduced by Hunt, attempts to minimise the difference between the unwrapped phase gradients and the wrapped values of the wrapped phase gradient, with the minimisation performed in a least-squares manner (Hunt, 1979), i.e., $p=2$ in Equation 3.25. In other words, the unweighted least squares method attempts to find the unwrapped phase solution that will minimise the following equation:

$$
\varepsilon^2 = \sum_{i=0}^{M-2} \sum_{j=0}^{N-2} \left( \nabla x \varphi(i, j) - \nabla x \psi(i, j) \right)^2 + \sum_{i=0}^{M-1} \sum_{j=0}^{N-2} \left( \nabla y \varphi(i, j) - \nabla y \psi(i, j) \right)^2
$$

Ghiglia et al. have simplified the above equation, and they showed that the above equation can be rewritten as: (Ghiglia and Pritt, 1998)

$$
(\varphi(i + 1, j) - 2\varphi(i, j) + \varphi(i - 1, j)) + (\varphi(i, j + 1) - 2\varphi(i, j) + \varphi(i, j - 1)) = \rho(i, j)
$$

where;

$$
\rho(i, j) = \left( \nabla x \psi(i, j) - \nabla x \psi(i - 1, j) \right) + \left( \nabla y \psi(i, j) - \nabla y \psi(i, j - 1) \right)
$$
Equation 3.32 represents a discrete version of Poisson’s equation and can be written in matrix notation by:

\[
P \vec{\varphi} = \vec{\rho}
\]  

(3.33)

where \(P\) represents the discrete Laplacian operation defined by the left-hand side of Equation 3.31, \(\vec{\varphi}\) and \(\vec{\rho}\) are one-dimensional vectors containing the values of the two-dimensional arrays \(\varphi(x, y)\) and \(\rho(x, y)\) respectively.

Unweighted least squared algorithms attempt to find the unwrapped phase solution by solving Poisson’s equation. Ghiglia et al. have proposed the use of Gauss-Seidel relaxation (Press et al., 1992) to find a solution for the unwrapped phase (Ghiglia and Pritt, 1998). In this method, initial values of zeros are given to the unwrapped phase solution. Then, the unwrapped phase solution is updated iteratively using Equation 3.34 until the difference or the error reaches a certain tolerance.

\[
\varphi(i, j) = \frac{(\varphi(i+1, j) + \varphi(i-1, j) + \varphi(i, j+1) + \varphi(i, j-1)) - \rho(i, j)}{4}
\]  

(3.34)

Ghiglia et al. have shown that using Gauss-Seidel relaxations is not practical due to its extremely slow convergence (Ghiglia and Pritt, 1998). On the other hand, they have proposed the use of multigrid methods (Briggs, 1987; Press et al., 1992) to speed up the convergence and achieve less error in a shorter time (Ghiglia and Pritt, 1998).

Another method of solving the unweighted least square phase unwrapping problem was also introduced by Ghiglia and Pritt (Ghiglia and Pritt, 1998), which was based on the use of the Fourier transform.
3.3.2 Weighted Least-squares

The unweighted least squares methods attempt to calculate the unwrapped phase solution that minimises Equation 3.25. These methods assign equal weights to all pixels in the wrapped phase map. If the wrapped phase contains residues and some corrupted areas, the whole solution will be affected and errors propagate through the entire solution space.

Weighted least squares algorithms are designed to overcome these problems. Each pixel in the wrapped phase map is assigned a certain weight. Noisy pixels and residues are assigned low weights to reduce their effect on the unwrapped solution. Weighted least squares methods attempt to find the unwrapped phase solution that minimises the following formula:

\[
\varepsilon^2 = \sum_{i=0}^{M-2} \sum_{j=0}^{N-1} U(i, j) \times \left| \Delta^x \phi(i, j) - \Delta^y \psi(i, j) \right|^2 + \sum_{i=0}^{M-1} \sum_{j=0}^{N-2} V(i, j) \times \left| \Delta^y \phi(i, j) - \Delta^y \psi(i, j) \right|^2 \tag{3.35}
\]

where; \( U(i, j) \) and \( V(i, j) \) are the gradient weights assigned for x-gradient and y-gradient respectively. These gradient weights are defined by:

\[
U(i, j) = \min(q_{i,j}^2, q_{i,j+1}^2) \tag{3.36}
\]

\[
V(i, j) = \min(q_{i,j+1}^2, q_{i,j}^2) \tag{3.37}
\]

where, \( q_{i,j} \) is the quality of the pixel \((i,j)\) calculated using one of the quality maps discussed earlier in this chapter.

Equation 3.35 can be rewritten as: (Ghiglia and Pritt, 1998)
Three-Dimensional Fourier Fringe Analysis and Phase Unwrapping

Chapter 3

\[ U(i, j)\Delta^x \varphi(i, j) - U(i - 1, j)\Delta^x \varphi(i - 1, j) + V(i, j)\Delta^y \varphi(i, j) - V(i, j - 1)\Delta^y \varphi(i, j - 1) = \sigma(i, j) \]  
(3.38)

where:

\[ \sigma(i, j) = \left[ U(i, j)\Delta^x \psi(i, j) - U(i - 1, j)\Delta^x \psi(i - 1, j) \right] + \left[ V(i, j)\Delta^y \psi(i, j) - V(i, j - 1)\Delta^y \psi(i, j - 1) \right] \]  
(3.39)

Using matrix notation, Equation 3.38 can be rewritten as:

\[ \mathbf{Z} \mathbf{\varphi} = \mathbf{\sigma} \]  
(3.40)

where \( \mathbf{Z} \) represents the discrete weighted Laplacian operation defined in the left-hand side of Equation 3.38 and \( \mathbf{\sigma} \) are one-dimensional vectors containing the values calculated from Equation 3.39.

Unlike the unweighted least squares methods, the weighted least squares method cannot be solved directly by means of a Fourier transform, therefore they must be solved iteratively.

The Gauss-Seidel relaxation used to solve the unweighted case can also be applied to the weighted least squares case, as proposed by Ghiglia and Pritt (Ghiglia and Pritt, 1998). The unwrapped phase solution can be calculated iteratively using the equation:

\[ \varphi(i, j) = \frac{\left[ U(i, j)\varphi(i + 1, j) + U(i - 1, j)\varphi(i - 1, j) + V(i, j)\varphi(i, j + 1) + V(i, j - 1)\varphi(i, j - 1) \right] - \sigma(i, j)}{v(i, j)} \]  
(3.41)

where \( v(i, j) \) is defined by:

\[ v(i, j) = U(i, j) + U(i - 1, j) + V(i, j) + V(i, j - 1) \]  
(3.42)
Because of the slow convergence of this algorithm, multigrid methods are used to speed up the convergence. (Ghiglia and Pritt, 1998)

Moreover, Ghiglia et al. have proposed two more methods to find the unwrapped phase solution. These methods are the Picard iteration method and the preconditioned conjugate gradient (PCG) method. The Picard method is simple and easy to implement, but the convergence is not guaranteed. In contrast, the PCG method has excellent convergence properties. (Ghiglia and Pritt, 1998)

### 3.3.3 \( L^p \)-Norm Method

A more advanced method, developed by Ghiglia and Romero, is the minimum \( L^p \)-norm method. This method finds the unwrapped phase solution that minimizes Equation 3.25 for an arbitrary value of \( p \), unlike the least squares method where \( p=2 \) (Ghiglia and Romero, 1996).

In the general case, Equation 3.25 can be simplified as: (Ghiglia and Pritt, 1998)

\[
\begin{align*}
\hat{U}(i, j)\Delta^x \varphi(i, j) - \hat{U}(i - 1, j)\Delta^x \varphi(i - 1, j) + \\
\hat{V}(i, j)\Delta^y \varphi(i, j) - \hat{V}(i, j - 1)\Delta^y \varphi(i, j - 1) &= \hat{\sigma}(i, j)
\end{align*}
\]  

(3.43)

where; \( \hat{U}(i, j) \) and \( \hat{V}(i, j) \) are the data dependent weights assigned for \( x \)-gradient and \( y \)-gradient respectively and they are defined by:

\[
\hat{U}(i, j) = \left| \Delta^x \varphi(i, j) - \tilde{\Delta}^x \psi(i, j) \right|^{p-2}
\]  

(3.44)

\[
\hat{V}(i, j) = \left| \Delta^y \varphi(i, j) - \tilde{\Delta}^y \psi(i, j) \right|^{p-2}
\]  

(3.45)

and \( \hat{\sigma}(i, j) \) is defined by:
\[
\hat{\sigma}(i, j) = \left( \hat{U}(i, j) \Delta^r \psi(i, j) - \hat{U}(i-1, j) \Delta^r \psi(i-1, j) \right) \\
+ \left( \hat{V}(i, j) \Delta^r \psi(i, j) - \hat{V}(i, j-1) \Delta^r \psi(i, j-1) \right)
\]  

(3.46)

Clearly from the equation, the L\(p\)-norm case is very similar to that of the case of weighted least squares. The only difference between the two cases is that of the definition of the weighted factors. In the weighted least squares method, the weighted factors are defined independently from a quality map, whereas in the L\(p\)-norm case these factors are extracted from the wrapped phase data itself. So, solving the L\(p\)-norm method can be carried out using the same algorithms used for the weighted least squares case (Ghiglia and Pritt, 1998)

### 3.3.4 Other global integration methods

Fernaro et al. have proposed a different global phase unwrapping algorithm. In their method, the phase unwrapping problem is introduced using Green’s formulation (Fornaro et al., 1995; Fornaro et al., 1996). Many researchers have proposed different phase unwrapping algorithms that estimate the parameters of a polynomial function that best matches the unwrapped solution by using regression (Slocumb and Kitchen, 1994; Schwarz, 2002; Karout et al., 2006).
3.4 Summary

This chapter presents a review of a number of two-dimensional phase unwrapping techniques that have been proposed in the past. Two main categories of phase unwrapping algorithms were discussed; local and global methods. Local phase unwrapping techniques attempt to place branch cuts, explicitly or implicitly, as barriers to prevent error propagation. On the other hand, global algorithms do not search for residues, nor do they place branch cuts to prevent error propagation. Instead, they attempt to find an unwrapped solution by estimating the unwrapped gradients, depending upon a minimisation approach. Both algorithms have their merits and demerits. In general terms, there is no unwrapper that is significantly better than any of the others and also there is no generic unwrapper that can overcome all phase unwrapping problems.

This chapter has also outlined a number of two-dimensional quality maps that aid phase unwrapping algorithms. Quality maps can be used by local or global phase unwrapping algorithms. The use of quality maps in designing a phase unwrapping algorithm will improve the performance of that algorithm.
References:


